Cole Shanks
54950860

$F(s) = \mathcal{L}\{f(t)\} = \int_0^\infty f(t) e^{-st}\,dt$

Eulers Formula:

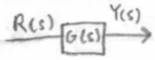
$e^{j\theta} = \cos\theta - j\sin\theta$

$\cos\theta = \dfrac{e^{j\theta} + e^{-j\theta}}{2}$

$\sin\theta = \dfrac{e^{j\theta} - e^{-j\theta}}{2j}$

$\text{sinc}(\theta) = \dfrac{\sin(\pi\theta)}{\pi\theta}$

Transfer Function:



$Y(s) = G(s) \cdot R(s)$

Laplace Transform:

$f(t)$	$F(s)$
$\delta(t)$	1
$u(t)$	$\frac{1}{s}$
$r(t)$	$\frac{1}{s^2}$
t^n	$\frac{n!}{s^{n+1}}$
e^{-at}	$\frac{1}{s+a}$
$\sin(at)$	$\frac{a}{s^2+a^2}$
$\cos(at)$	$\frac{s}{s^2+a^2}$
te^{-at}	$\frac{1}{(s+a)^2}$
$e^{-at}\sin(bt)$	$\frac{b}{(s+a)^2+b^2}$
$e^{-at}\cos(bt)$	$\frac{s+a}{(s+a)^2+b^2}$
$t^n f(t)$	$(-1)^n \frac{d}{ds^n}(F(s))$

Properties:

① Time Delay, $f(t-T)\,u(t-T) \longleftrightarrow e^{-Ts}F(s)$

② Differentiation,

$f''(t) + f'(t) + f(t)$

$\updownarrow$

$\boxed{s^2F(s) - sf(0) - f'(0)} + \boxed{sF(s) - f(0)} + \boxed{F(s)}$

③ Final Value Thrm, "if all the poles of $sF(s)$ are in the LHP with possibly one simple pole at the origin"

$\boxed{\lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s)}$

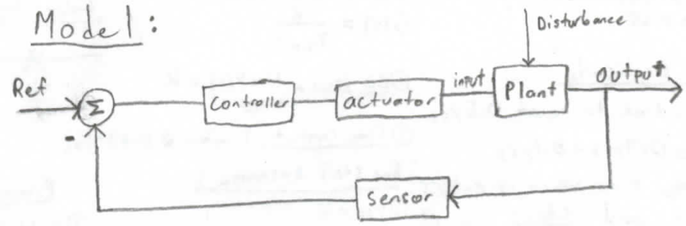
④ Initial Value Thrm, "if the limit exists. ie: no requirements"

$\boxed{\lim_{t\to 0^+} f(t) = \lim_{s\to\infty} sF(s)}$

⑤ Frequency shift,

$e^{-at}f(t) \longleftrightarrow F(s+a)$

Model:

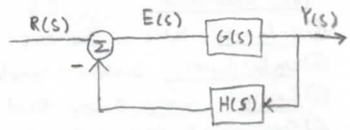


Similarities:

① Capacitor ~ Mass ~ Inertia

② Inductor ~ Spring ~ Rotational spring

③ Resistor ~ Damper ~ Friction



Definitions:

① $G(s)$ = Forward Transfer Function

② $H(s)$ = Feedback Transfer Function

③ $G(s)H(s)$ = Open-Loop TF

④ $\frac{C(s)}{R(s)}$ = Closed-loop TF

⑤ $\frac{C(s)}{E(s)}$ = Feed-forward TF

State-Space Modeling:

$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = [A]\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + [B][u]$

$[y] = [C]\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + [D][u]$

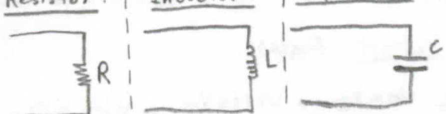
$T(s) = C \cdot (sI - A)^{-1} \cdot B + D$

Black's Formula:

$Y(s) = \dfrac{G(s)}{1 + G(s)H(s)} \cdot R(s)$

*if Positive Feedback, $\oplus \to \ominus$

Electrical Elements:



Resistor: $Z = R$

Inductor: $Z = sL$

Capacitor: $Z = \frac{1}{sC}$

Stability:

① STABLE → All poles in LHP

② Marginally STABLE
- No poles in RHP
- All poles on $j\omega$ axis have multiplicity 1

③ UNSTABLE → Poles in RHP

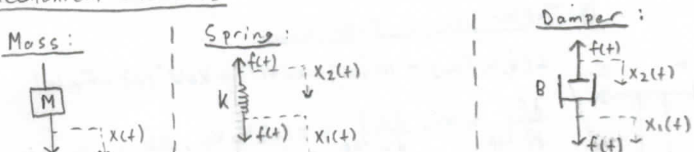
Tricks:

① 1st Order, all coefficients have the same sign then all poles in LHP

② 2nd order, all coefficients have the same sign then all poles in LHP

③ 3rd order, if even one coefficient has a different sign you will have roots in RHP

Mechanical Elements:



Mass:

$f(t) = Mx''(t)$

$\updownarrow$

$F(s) = Ms^2X(s)$

Spring:

$f(t) = k[x_1(t) - x_2(t)]$

$\updownarrow$

$F(s) = k[X_1(s) - X_2(s)]$

Damper:

$f(t) = B[x_1'(t) - x_2'(t)]$

$\updownarrow$

$F(s) = Bs[X_1(s) - X_2(s)]$

Routh-Array:

s^4	a_n	a_{n-2}	a_{n-4}
s^3	a_{n-1}	a_{n-3}	
s^2	b_1	b_2	
s^1	c_1		
s^0	d_1		

$b_1 = \dfrac{(a_{n-1})(a_{n-2}) - (a_n)(a_{n-3})}{(a_{n-1})}$

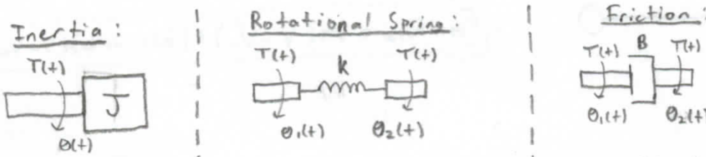
$b_2 = \dfrac{(a_{n-1})(a_{n-4}) - (a_n)(a_{n-5})}{(a_{n-1})}$

$c_1 = \dfrac{(b_1)(a_{n-3}) - (a_{n-1})(b_2)}{(b_1)}$

$c_2 = \dfrac{(b_1)(a_{n-5}) - (a_{n-1})(b_3)}{(b_1)}$

"The number of sign changes in the first column is equal to the number of roots in the RHP"

Rotational Elements:



Inertia:

$T(t) = J\theta''(t)$

$\updownarrow$

$T(s) = Js^2\theta(s)$

Rotational Spring:

$T(t) = k[\theta_1(t) - \theta_2(t)]$

$\updownarrow$

$T(s) = k[\theta_1(s) - \theta_2(s)]$

Friction:

$T(t) = B[\theta_1'(t) - \theta_2'(t)]$

$\updownarrow$

$T(s) = Bs[\theta_1(s) - \theta_2(s)]$

Ex: "Inverse of a 3x3 Matrix"

$A = \begin{bmatrix} 0 & 0 & 1 \\ 2 & -1 & 3 \\ 1 & 1 & 4 \end{bmatrix} \quad \det(A) = 3$

$\begin{bmatrix} +\begin{vmatrix} -1 & 3 \\ 1 & 4 \end{vmatrix} & -\begin{vmatrix} 2 & 3 \\ 1 & 4 \end{vmatrix} & +\begin{vmatrix} 2 & -1 \\ 1 & 1 \end{vmatrix} \\ -\begin{vmatrix} 0 & 1 \\ 1 & 4 \end{vmatrix} & +\begin{vmatrix} 0 & 1 \\ 1 & 4 \end{vmatrix} & -\begin{vmatrix} 0 & 0 \\ 1 & 1 \end{vmatrix} \\ +\begin{vmatrix} 0 & 1 \\ -1 & 3 \end{vmatrix} & -\begin{vmatrix} 0 & 1 \\ 2 & 3 \end{vmatrix} & +\begin{vmatrix} 0 & 0 \\ 2 & -1 \end{vmatrix} \end{bmatrix}$

$= \begin{bmatrix} -7 & -5 & 3 \\ 1 & -1 & 0 \\ 1 & 2 & 0 \end{bmatrix}$ *Flip over Diagonal

Ex: "Inverse of a 2x2"

$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

$A^{-1} = \dfrac{1}{\det(A)}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

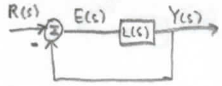
Time Response:

$y(t) = \underbrace{y_t(t)}_{\text{Transient}} + \underbrace{y_{ss}(t)}_{\text{Steady-state}}$

① Transient $= 0$ as $t \to \infty$

② Steady-state $= \lim_{t\to\infty} y(t)$

Steady-State Error: $e_{ss} = \lim_{t\to\infty} e(t) = \lim_{s\to 0} sE(s) = \lim_{s\to 0} s \cdot \frac{1}{1+L(s)} \cdot R(s)$



*Unity feedback = 1 on the bottom

Error Constants:

Step-Error: $K_p = \lim_{s\to 0} L(s)$

Ramp-Error: $K_v = \lim_{s\to 0} s \cdot L(s)$

Parabolic-Error: $K_a = \lim_{s\to 0} s^2 \cdot L(s)$

ess for Inputs:

① $r(t) = Ru(t)$, $e_{ss} = \frac{R}{1+K_p}$

② $r(t) = Rtu(t)$, $e_{ss} = \frac{R}{K_v}$

③ $r(t) = \frac{Rt^2}{2}u(t)$, $e_{ss} = \frac{R}{K_a}$

Damping Ratios:

① Undamped, $z = 0$

② Underdamped, $0 < z < 1$

③ Critically Damped, $z = 1$

④ Overdamped, $z > 1$

Accurate Tracking ($e_{ss} = 0$):

$K_p = K_v = K_a = \infty$

Time Values Graphically:

① Delay time, time to reach $0.5y_{ss}$

② Rise time, $0.9y_{ss} - 0.1y_{ss}$

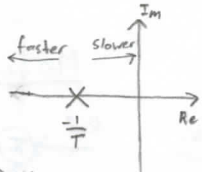
③ Settling time, Time when it enters percent box and stays in it

④ Percent overshoot, $\frac{y_{max} - y_{ss}}{y_{ss}} \cdot 100\%$

⑤ Peak Time, time to reach y_{max}

1st Order System:

$G(s) = \frac{K}{Ts+1}$



① DC Gain, $\lim_{t\to\infty} y(t) = K$

② Time Constant, T when @ $0.63 \cdot y_{ss}$

For Step Response:

① $y_{ss} = K$

② y_{max}, T_p, PO = UNDEFINED

③ Delay time $= 0.7T$

④ Rise time $= 2.2T$

⑤ Settling time,

$2\% = 4T$

$5\% = 3T$

2nd Order System:

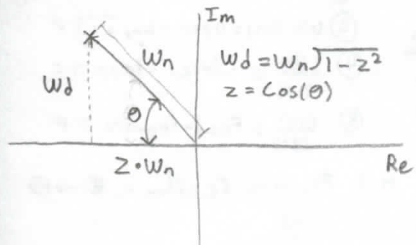
$G(s) = \frac{K}{s^2 + 2zW_n \cdot s + W_n^2}$

① Damping Ratio, 'z'

② Undamped Natural Frequency, 'W_n'

Location of Pole gives you Everything:

[2nd Order] $*0 < z < 1*$



$W_d = W_n\sqrt{1-z^2}$

$z = \cos(\theta)$

Properties of 2nd Order Systems:

① Settling time, $2\% = \frac{4}{z \cdot W_n}$ $5\% = \frac{3}{z \cdot W_n}$ $*T = \frac{1}{z \cdot W_n}$

② Peak time, $\frac{\pi}{W_d}$

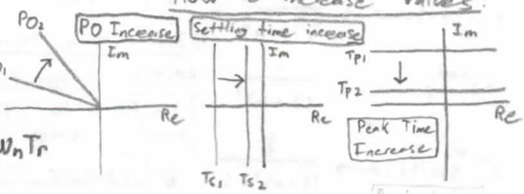
③ Peak Value, $1 + e^{-z\cdot\pi/\sqrt{1-z^2}}$

④ Percent Overshoot, $100e^{-z\cdot\pi/\sqrt{1-z^2}}$

⑤ Rise Time, $1.76z^3 - 0.417z^2 + 1.039z + 1 = W_n T_r$

⑥ y_{max}, $1 + e^{\frac{-z\cdot\pi}{\sqrt{1-z^2}}}$

How to increase Values:



Pole Locations:

① Undamped, Poles on $j\omega$ axis

② Underdamped, Distinct Complex Poles

③ Critically damped, Double Real Poles

④ Overdamped, Distinct Real Poles

$PO = 100e^{-\pi/\tan\theta}$

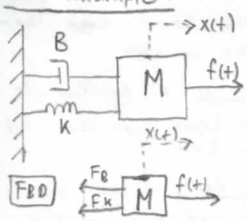
Dominant Poles

$G(s) = \frac{10}{(s+1)(s+10)} \approx \frac{1}{s+1}$ (5-10x Far away)

Complex Formula:

$$z = \frac{\ln(\frac{PO}{100})}{\sqrt{\pi^2 + (\ln(\frac{PO}{100}))^2}}$$

FBD Example:



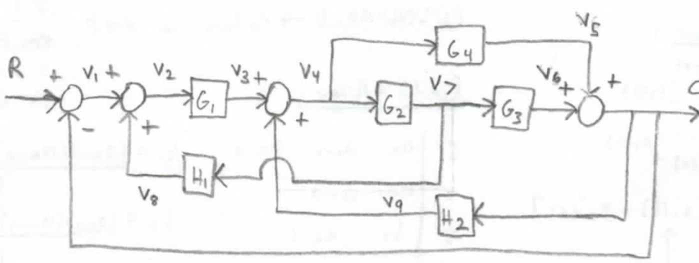
$F_B = (-B)(\dot{x}_{LEFT} - \dot{x}_{RIGHT})$

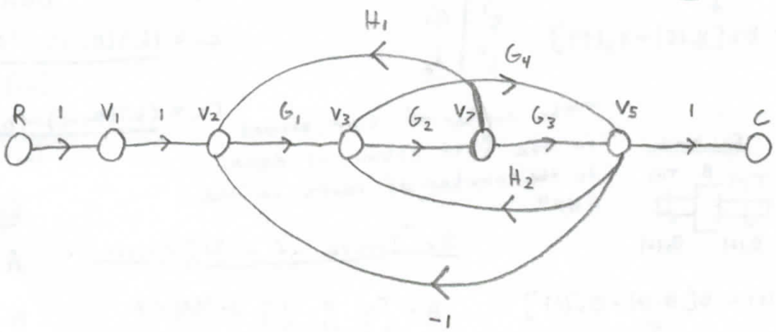
$F_K = (-K)(x_{LEFT} - x_{RIGHT})$

$M\ddot{x} = f(t) - F_B - F_K$

$\frac{X(s)}{F(s)} = \frac{1}{Ms^2 + Bs + K}$

Signal Flow Example:





Ex: "Linearize"

$m\dot{v}(t) + k_f v(t) + k_a v^2(t) = F_m(t)$ @ OP → $V_0 = 30$

① Input: $v(t)$ output: $F_m(t)$

② Operating Point: $V_0 = 30 \to v(t) = V_0 \to \dot{v}(t) = 0$

$\therefore \cancel{m\dot{v}(t)} + k_f v(t) + k_a v^2(t) = F_m(t)$

$k_f V_0 + k_a V_0^2 = F_{m0}$

③ Taylor Series Expansion:

$f(\dot{v}, v, F_m) = m\dot{v}(t) + k_f v(t) + k_a v^2(t) - F_m(t)$

$\frac{df}{d\dot{v}}\Big|_{op} = m$, $\frac{df}{dv}\Big|_{op} = k_f + 2k_a V_0$, $\frac{df}{dF_m}\Big|_{op} = -1$

$\therefore f(\dot{v}, v, F_m) = f(\dot{V}_0, V_0, F_{m0}) + m\delta\dot{v} + (k_f + 2k_a V_0)\delta v - \delta F_m$

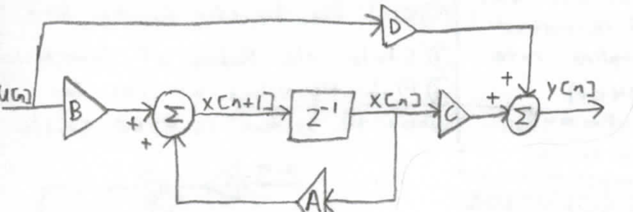
$0 = 0 + m\delta\dot{v} + (k_f + 2k_a V_0)\delta v - \delta F_m$

$\therefore \delta F_m = m\delta\dot{v} + (k_f + 2k_a V_0)\delta v$

$\boxed{F_m - F_{m0} = m(\dot{v} - \dot{V}_0) + (k_f + 2k_a V_0)(v - V_0)}$

Cole Shanks
54950860

DT System State-Space LTI Model



$$x[n+1] = Ax[n] + Bu[n]$$
$$y[n] = Cx[n] + Du[n]$$

State Transition Matrix

$$x[n] = \underbrace{A^n x_0}_{ZIR} + \underbrace{\sum_{k=0}^{n-1} A^{n-1-k} Bu[k]}_{ZSR}$$

$$y[n] = \underbrace{CA^n x_0}_{ZIR} + \underbrace{\sum_{k=0}^{n-1} CA^{n-1-k} Bu[k] + Du[n]}_{ZSR}$$

$$Y(z) = C(zI-A)^{-1}x_0 + \underbrace{[C(zI-A)^{-1}B + D]}_{H(z)}U(z)$$

How to Compute State Transition Matrix?

① $\det(A-\lambda I) = 0 \rightarrow Av_i = \lambda_i v_i$

$$A = [v_1, v_2 \dots v_n]\begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & & \lambda_n \end{bmatrix}[v_1, v_2, \dots v_n]^{-1} = TDT^{-1}$$

State Transition matrix

CT $\rightarrow$ $$e^{At} = T\begin{bmatrix} e^{\lambda_1 t} & 0 & \cdots & 0 \\ 0 & e^{\lambda_2 t} & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & & e^{\lambda_N t} \end{bmatrix}T^{-1}$$

DT $\rightarrow$ $$A^n = T\begin{bmatrix} \lambda_1^n & 0 & \cdots & 0 \\ 0 & \lambda_2^n & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & & \lambda_N^n \end{bmatrix}T^{-1}$$

② Caley-Hamilton Thereom

State Transition matrix

CT $\rightarrow$ $e^{At} = \sum_{k=0}^{N-1} c_k A^k$ where $\forall i \in [1, N]$

$e^{\lambda i t} = \sum_{k=0}^{n-1} c_k{}^{\lambda i k}$

DT $\rightarrow$ $A^n = \sum_{k=0}^{N-1} c_k A^k$ where $\forall i \in [1, N]$

$\lambda_i^n = \sum_{k=0}^{N-1} c_k \lambda_i^k$

Frequency Response

$x(t) = e^{j\omega t} \rightarrow$ CT $\rightarrow y(t) = |H(\omega)|e^{j(\omega t + \angle H(\omega))}$

$x[n] = e^{j\omega n} \rightarrow$ DT $\rightarrow y[n] = |H(\omega)|e^{j(\omega n + \angle H(\omega))}$

Ex: "Matrix Multiplication"

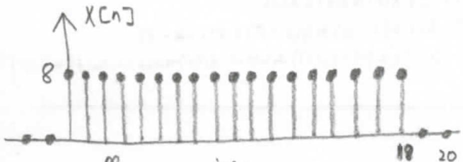
$$A = \begin{bmatrix} 3 & -1 & 0 \\ 2 & 5 & 1 \\ -7 & 1 & 3 \end{bmatrix} \quad B = \begin{bmatrix} 6 & -1 & 0 \\ 0 & 1 & -2 \\ 3 & -8 & 1 \end{bmatrix}$$

$$AB = \begin{bmatrix} (3)(6)+(-1)(0)+(0)(3) & (3)(-1)+(-1)(1)+(0)(-8) & \cdots \\ (2)(6)+(5)(0)+(1)(3) & (2)(-1)+(5)(1)+(1)(-8) & \cdots \\ (-7)(6)+(1)(0)+(3)(3) & (-7)(-1)+(1)(1)+(3)(-8) & \end{bmatrix}$$

Inverse of a 2x2 Matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad A^{-1} = \frac{1}{\det(A)} \cdot \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Ex: "DT Fourier Transform"



$$X(\omega) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$$

$$= \sum_{n=0}^{18} (8)e^{-j\omega n} \longrightarrow \frac{8(1-e^{-j\omega n(19)})}{1-e^{-j\omega n}}$$

$$= \frac{8e^{-j\omega n(19/2)}[e^{j\omega n(19/2)} - e^{-j\omega n(19/2)}]}{e^{-j\omega(1/2)}[e^{j\omega(1/2)} - e^{-j\omega(1/2)}]}$$

$$= 8e^{-j\omega(9)} \cdot \frac{\sin(19\omega/2)}{\sin(\omega/2)}$$

Ex: "Inverse of a 3x3 Matrix"

$$A = \begin{bmatrix} 0 & 0 & 1 \\ 2 & -1 & 3 \\ 1 & 1 & 4 \end{bmatrix} \quad \det(A) = 3$$

$$\begin{bmatrix} +\begin{vmatrix} -1 & 3 \\ 1 & 4 \end{vmatrix} & -\begin{vmatrix} 2 & 3 \\ 1 & 4 \end{vmatrix} & +\begin{vmatrix} 2 & -1 \\ 1 & 1 \end{vmatrix} \\ -\begin{vmatrix} 0 & 1 \\ 1 & 4 \end{vmatrix} & +\begin{vmatrix} 0 & 1 \\ 1 & 4 \end{vmatrix} & -\begin{vmatrix} 0 & 0 \\ 1 & 1 \end{vmatrix} \\ +\begin{vmatrix} 0 & 1 \\ -1 & 3 \end{vmatrix} & -\begin{vmatrix} 0 & 1 \\ 2 & 3 \end{vmatrix} & +\begin{vmatrix} 0 & 0 \\ 2 & -1 \end{vmatrix} \end{bmatrix}$$

$$= \begin{bmatrix} -7 & -5 & 3 \\ 1 & -1 & 0 \\ 1 & 2 & 0 \end{bmatrix}$$

* Flip over diagonal

$$A^{-1} = \frac{1}{3}\begin{bmatrix} -7 & 1 & 1 \\ -5 & -1 & 2 \\ 3 & 0 & 0 \end{bmatrix}$$

CT: $e^{\lambda_1 t} = \alpha_0 + (\lambda_1)\alpha_1$

$e^{\lambda_2 t} = \alpha_0 + (\lambda_2)\alpha_1$

DT: $\lambda_1^n = \alpha_0 + (\lambda_1)\alpha_1$

$\lambda_2^n = \alpha_0 + (\lambda_2)\alpha_1$

Ex: "Cayley Hamilton Thereom"

$$A = \begin{bmatrix} 0 & 7 \\ -\frac{72}{7} & 17 \end{bmatrix} \rightarrow \lambda^2 - 17\lambda + 72 = 0$$

$(\lambda-9)(\lambda-8) \therefore \lambda_1 = 8$, $\lambda_2 = 9$

$$e^{At} = A^n = \alpha_0 I + \alpha_1 A$$

$$f(A) = \alpha_0\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \alpha_1\begin{bmatrix} 0 & 7 \\ -\frac{72}{7} & 17 \end{bmatrix}$$

For Bigger Matrices:

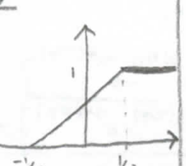
$$e^{At} = \alpha_0 I + \alpha_1 A + \alpha_2 A^2$$
$$A^n = \alpha_0 I + \alpha_1 A + \alpha_2 A^2$$

Transfer function from State Diagram

$$H(s) = \frac{b_0 s^N + b_1 s^{N-1} \dots + b_N}{s^N + a_1 s^{N-1} \dots + a_n}$$

Ex: "Differentiation Property"

$$x(t) = \begin{cases} 0, & t < -1/2 \\ t+\frac{1}{2}, & -\frac{1}{2} < t < \frac{1}{2} \\ 1, & t > 1/2 \end{cases}$$



$$x(t) = r(t+\tfrac{1}{2}) - r(t-\tfrac{1}{2})$$

$$\frac{dx(t)}{dt} = u(t+\tfrac{1}{2}) - u(t-\tfrac{1}{2})$$

$$\frac{d^2}{dt^2}x(t) = \delta(t+\tfrac{1}{2}) - \delta(t-\tfrac{1}{2})$$

$\updownarrow$ FT

$$(j\omega)^2 X(\omega) = e^{j(\frac{\omega}{2})} - e^{-j(\frac{\omega}{2})}$$

$$X(\omega) = \frac{2j\sin(\omega/2)}{(j\omega)^2}$$

$$X(\omega) = \frac{2\sin(\omega/2)}{j\omega^2}$$

Ex: "DT Response"

"Consider a filter with difference equation $y[n] - \frac{1}{4}y[n-2] = x[n] + \frac{1}{2}x[n-1]$"

Find
① Frequency and impulse response
② Output when input is $x[n] = \cos(\frac{\pi n}{2})$

SOLUTION

$$Y(z) - \frac{1}{4}z^{-2}Y(z) = X(z) + \frac{1}{2}z^{-1}X(z)$$

$$Y(z)[1 - \tfrac{1}{4}z^{-2}] = X(z)[1 + \tfrac{1}{2}z^{-1}]$$

$$H(z) = \frac{1+\frac{1}{2}z^{-1}}{1-\frac{1}{4}z^{-2}} = \frac{1}{1-\frac{1}{2}z^{-1}}$$

FREQ Response:

$$H(\omega) = \frac{1}{1-\frac{1}{2}e^{-j\omega}}$$

IMPULSE RESPONSE:

$$H[n] = Z^{-1}\{H(z)\} = (\tfrac{1}{2})^n u[n]$$

$$H(\tfrac{\pi}{2}) = \frac{1}{1-\frac{1}{2}e^{-j(\pi/2)}} = \frac{1 \leftarrow z_1}{1+\frac{1}{2}j \leftarrow z_2}$$

$$|H(\tfrac{\pi}{2})| = \frac{1}{\sqrt{1+(\frac{1}{2})^2}} \quad \angle H(\tfrac{\pi}{2}) = 0 - \tan^{-1}(\tfrac{1/2}{1})$$

DT Causalities

① Causal, attach $u[n]$ ie: $(A[n]+B[n])u[n]$

② Two-sided, attach both ie: $A[n]u[n] + B[n](-u[-n-1])$

③ Anti-causal, attach $(-u[-n-1])$ ie: $(A[n]+B[n])(-u[-n-1])$

Cole Shanks
54950860

Ex: "Discrete Time Convolution"

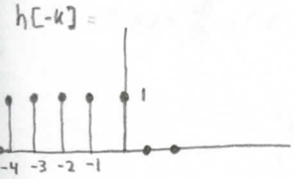
"The impulse responce of a discrete LTI system is given by $h[n] = u[n] - u[n-5]$. Given that the input to the system is given by $x[n] = 3(u[n] - u[n-6])$. Find $y[n]$.

SOLUTION:

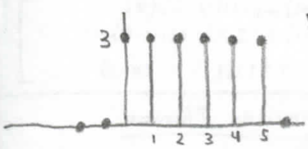
$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

$h[n] = u[n] - u[n-5]$

$h[-k]$



$x[n] = 3(u[n] - u[n-6])$



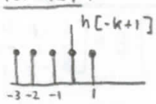
Steps:

1. For (+) values of n, shift $h[-k]$ to the right
2. For (-) values shift left
3. For $n=0$, use $h[-k]$
4. Sum over all k: $x[k] \cdot h[-k+n]$ and that is the value for $y[n]$

For $n=0$:

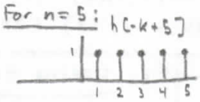
$y[0] = (3)(1) = 3$

For $n=1$: $h[-k+1]$



$y[1] = (3)(1) + (3)(1) = 6$

For $n=5$: $h[-k+5]$



$y[5] = (3)(1) + (3)(1) + (3)(1) + (3)(1) + (3)(1) = 15$

Ex: "Aliasing" "A continuous-time signal

$x(t) = 7\cos(23\pi t) + 7\sin(24\pi t) + 4\cos(46\pi t + \pi/9)$ is sampled at 24Hz. Determine $w(t)$ reconstructed using an ideal interpolator at a sampling rate of $1/24$ s

SOLUTION:

$f_s = 24$

$W_s = 48\pi$

$T_s = 1/24$

1. $7\cos(23\pi t)$: $2W_m = 46\pi < 48\pi$ ∴ Reconstructed completely
2. $7\sin(24\pi t)$: $2W_m = 48\pi = 48\pi$ ∴ DC value because sine
3. $4\cos(46\pi t + \pi/9)$: $2W_m = 92\pi > 48\pi$ ∴ Aliasing

$4\cos(46\pi t - kt(2\pi(1/T_s)) + \pi/9)$

"Choose integer value for k that puts it within nyquist range ie: 2"

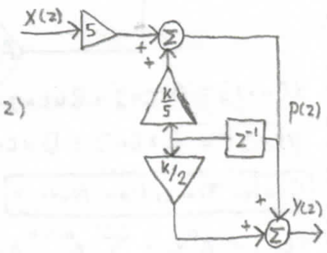
$4\cos(46\pi t - 48\pi t + \pi/9)$

$= 4\cos(-2\pi t + \pi/9)$

$\therefore W(t) = 7\cos(23\pi t) + 4\cos(-2\pi t + \pi/9)$

Ex: "Block Diagram"

1. Find the transfer function $H(z)$
2. State the Radius of Convergence
3. Find the values of $|k|$ for which the system is BIBO Stable



SOLUTION:

1. $p(z) = 5X(z) + \frac{k}{5} z^{-1} p(z)$

$\therefore p(z) = \dfrac{5X(z)}{1 - \frac{k}{5} z^{-1}}$

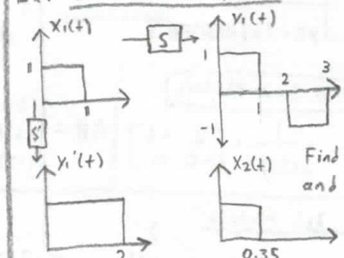
$Y(z) = p(z) + \frac{k}{2} z^{-1} p(z)$

$Y(z) = \dfrac{5X(z)}{1 - \frac{k}{5} z^{-1}} + \dfrac{k}{2} \cdot \dfrac{5X(z) z^{-1}}{1 - \frac{k}{5} z^{-1}}$

$\dfrac{Y(z)}{X(z)} = \dfrac{5}{1 - \frac{k}{5} z^{-1}} + \dfrac{k \cdot 5 \cdot z^{-1}}{2(1 - \frac{k}{5} z^{-1})}$ → $H(z) = \dfrac{5(z + \frac{k}{2})}{z - \frac{k}{5}}$

2. ROC: $(\frac{k}{5}, \infty)$
3. BIBO: $(0, 5)$

Ex: "LTI System Property"



Find $y_2(t)$ and $y_2'(t)$

SOLUTION:

$x_1(t) = u(t) - u(t-1)$

$y_1(t) = x_1(t) - x_1(t+2)$

$y_1'(t) = x_1(t) + x_1(t-1)$

$h(t) = \delta(t) - \delta(t-2)$

$h'(t) = \delta(t) + \delta(t-1)$

→ $x_2(t) = u(t) - u(t-0.35)$

$\therefore y_2(t) = u(t) - u(t-0.35) - u(t-2) + u(t-0.35-2)$

$y_2'(t) = u(t) - u(t-0.35) + u(t-1) - u(t-0.35-1)$

Convolution Properties:

(i) $x(t) * \delta(t) = x(t)$

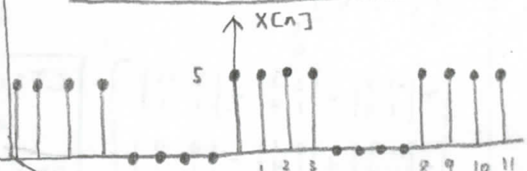
(ii) $x(t-\alpha) * h(t-\beta) = x(t-\alpha-\beta)$

(iii) $(x_1(t) + x_2(t)) * h(t) = x_1(t) * h(t) + x_2(t) * h(t)$

Ex: "Different Causalities"

Impulse Response:	Laplace Transform:	ROC:	BIBO:
$8e^{-6t}u(t) + 15e^{2t}u(t)$	$\frac{8}{s+6} + \frac{15}{s-2}$	(2, INF)	NO
$-7e^{-4t}u(-t) - 15e^{2t}u(-t)$	$\frac{-7}{(-1)(s+4)} - \frac{15}{(-1)(s-2)}$	(-INF, -4)	NO
$19e^{-6t}u(t) - 13e^{2t}u(-t)$	$\frac{19}{s+6} - \frac{13}{(-1)(s-2)}$	(-6, 2)	YES

Ex: "DT Fourier Series Coefficients"



Find the fourier coefficients X_k for $k \neq 0$.

SOLUTION:

$N = 8$

$W_0 = \frac{2\pi}{N} = \frac{\pi}{4}$

$X_k = \frac{1}{N} \sum_{n=0}^{N-1} x[n] e^{-jk(\pi/4)n}$

$X_k = \frac{1}{8} \sum_{n=0}^{3} (5) e^{-jk(\pi/4)n} + \varnothing$

Known:

$a = 5$

$r = e^{-jk(\pi/4)}$

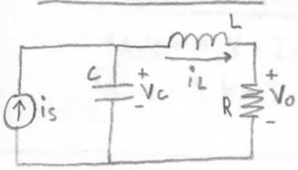
$N = 3$

$\sum_{n=0}^{N} ar^n = \dfrac{5(1 - e^{-jk(\pi/4) \cdot 4})}{1 - e^{-jk(\pi/4)}}$

$= \dfrac{5 \cdot e^{-jk(\pi/2)} \cdot [e^{jk(\pi/2)} - e^{-jk(\pi/2)}]}{e^{-jk(\pi/8)} \cdot [e^{jk(\pi/8)} - e^{-jk(\pi/8)}]}$

$X_k = \frac{5}{8} e^{-jk(3\pi/8)} \cdot \dfrac{\sin(k\pi/2)}{\sin(k\pi/8)}$

Ex: "State Space Circuit"



Given:

$R = 77\Omega$

$L = 7H$

$C = 1/196 F$

Known:

$i = C\dfrac{dV_c}{dt}$

$V = L\dfrac{di_L}{dt}$

Find:

1. State space Representation [A, B, C, D]
2. Transfer function H(s)
3. State Transition Matrix $\Phi(t)$
4. Time domain response with zero-input and initial conditions $x_1(0) = 4$, $x_2(0) = 5$

SOLUTION:

KCl: $i_s = C\dfrac{dV_c}{dt} + i_L$ KVL: $V_c - L\dfrac{di_L}{dt} - R \cdot i_L = 0$

$C\dfrac{dV_c}{dt} = i_s - i_L$ $L\dfrac{di_L}{dt} = V_c - R \cdot i_L$

$\boxed{\dot{V}_c = \frac{1}{C} i_s - \frac{1}{C} i_L}$ $\boxed{\dot{i}_L = \frac{1}{L} V_c - \frac{R}{L} i_L}$

KCl: $\dfrac{V_0}{R} = i_L \rightarrow \boxed{V_0 = R \cdot i_L}$

$$\begin{bmatrix} \dot{V}_c \\ \dot{i}_L \end{bmatrix} = \overset{A}{\begin{bmatrix} 0 & -1/C \\ 1/L & -R/L \end{bmatrix}} \begin{bmatrix} V_c \\ i_L \end{bmatrix} + \overset{B}{\begin{bmatrix} 1/C \\ 0 \end{bmatrix}} i_s$$

$$[V_0] = \overset{C}{[0 \quad R]} \begin{bmatrix} V_c \\ i_L \end{bmatrix} + \overset{D}{[0]} i_s$$

2. KCl:

$i_s = V_1(s \cdot C) + \dfrac{V_1}{s \cdot L + R}$

$\dfrac{V_0}{i_s} = \dfrac{R}{CLs^2 + CRs + 1}$

3.

$\vec{x} = \begin{bmatrix} V_c \\ i_L \end{bmatrix}$ $A = \begin{bmatrix} 0 & -196 \\ 1/7 & -11 \end{bmatrix}$

$\det(A - \lambda I) = \begin{vmatrix} 0-\lambda & -196 \\ 1/7 & -11-\lambda \end{vmatrix}$

$(-\lambda)(-11-\lambda) + \frac{196}{7} = 0$

$\lambda^2 + 11\lambda + \frac{196}{7} = 0$

$\lambda_1 = -4$, $\lambda_2 = -7$

For $\lambda = -4$:

$A = \begin{bmatrix} 4 & -196 \\ 1/7 & -7 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\frac{1}{7} a_1 - 7a_2 = 0$

$\frac{1}{7} a_1 = 7a_2$

$a_1 = 49a_2$

$a^{(1)} = \begin{pmatrix} 49 \\ 1 \end{pmatrix}$

For $\lambda = -7$:

$A = \begin{bmatrix} 7 & -196 \\ 1/7 & -4 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\frac{1}{7} a_1 = 4a_2$

$a_1 = 28a_2$

$a^{(2)} = \begin{pmatrix} 28 \\ 1 \end{pmatrix}$

$e^{At} = \begin{bmatrix} 49 & 28 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} e^{-4t} & 0 \\ 0 & e^{-7t} \end{bmatrix} \begin{bmatrix} 49 & 28 \\ 1 & 1 \end{bmatrix}^{-1}$

4.

$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \left(\begin{bmatrix} 49 & 28 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} e^{-4t} & 0 \\ 0 & e^{-7t} \end{bmatrix} \begin{bmatrix} 49 & 28 \\ 1 & 1 \end{bmatrix}^{-1} \right) \begin{bmatrix} 4 \\ 5 \end{bmatrix}$

X_0 (I.C's)

Ex: "Diagonalization of a 3x3 Matrix"

$A = \begin{pmatrix} 2 & 1 & 3 \\ 1 & 2 & 3 \\ 3 & 3 & 20 \end{pmatrix} \rightarrow \det(A - \lambda I) = \begin{vmatrix} 2-\lambda & 1 & 3 \\ 1 & 2-\lambda & 3 \\ 3 & 3 & 20-\lambda \end{vmatrix}$

$= (2-\lambda)[(2-\lambda)(20-\lambda) - 9] - (1)[(20-\lambda) - 9] + (3)[3 - 3(2-\lambda)]$

$= \lambda^3 - 24\lambda^2 + 65\lambda - 42$

1. Guess root $(\lambda - 1)$

$= (\lambda - 1)(\lambda^2 - 23\lambda + 42)$

$= (\lambda - 1)(\lambda - 2)(\lambda - 21)$

For $\lambda = 2$: *reduce to free variable*

$A = \begin{pmatrix} 0 & 1 & 3 \\ 1 & 0 & 3 \\ 3 & 3 & 18 \end{pmatrix} \xrightarrow{RREF} \begin{pmatrix} 1 & 0 & 3 \\ 0 & 1 & 3 \\ 0 & 0 & 9 \end{pmatrix}$

$\begin{pmatrix} 1 & 0 & 3 \\ 0 & 1 & 3 \\ 0 & 0 & 9 \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$a_1 = -3a_3$

$a_2 = -3a_3$

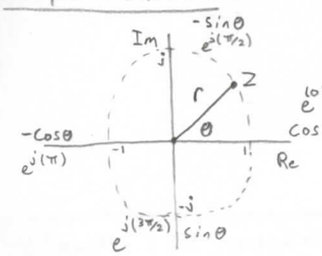
$a_3 = a_3$

$a^{(2)} = \begin{pmatrix} -3 \\ -3 \\ 1 \end{pmatrix}$

Cole Shanks
54950860

MT1 Material

Complex Numbers:



Cartesian Form: $Z = x + jy$

Rectangular Form: $Z = (x, y)$

Polar Form: $Z = r\angle\theta$

Exponential Form: $Z = re^{j\theta}$

Euler Identity: $e^{j\theta} = \cos\theta + j\sin\theta$

Classifications of Signals

① **Deterministic** vs. **Stochastic**: Represented by a formula / Probabilistic Distribution

② **Continuous** vs. **Discrete**: Domain or range / Domain or range

③ **Even** vs. **ODD**: $x(t) = x(-t)$ / $x(t) = -x(-t)$

④ **Periodic** vs. **Aperiodic**: $x(t + kT) = x(t)$ / No repetition

⑤ **Finite E_x/P_x** vs. **Infinite E_x/P_x**: $E_x < \infty$, $P_x < \infty$ / $E_x \to \infty$, $P_x \to \infty$

⑥ **Causal** vs. **Acausal**: $x(t) = 0;\ t < 0$ / Both sides of time (2 sided)

Anticausal: $x(t) = 0,\ t > 0$

Even/Odd Signals

$y(t) = y_e(t) + y_o(t)$

$y_e(t) = \frac{1}{2}[y(t) + y(-t)]$

$y_o(t) = \frac{1}{2}[y(t) - y(-t)]$

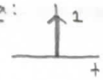
Energy / Power

$$E_x = \int_{-\infty}^{\infty} |x(t)|^2 dt$$

$$P_x = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} |x(t)|^2 dt$$

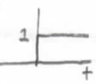
Basic Signals

Dirac Delta: $\delta(t)$

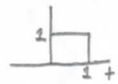


Unit Ramp: $r(t)$ $\begin{cases} t, & t > 0 \\ 0, & t < 0 \end{cases}$

Unit Step: $u(t)$ $\begin{cases} 1, & t > 0 \\ 0, & t < 0 \end{cases}$

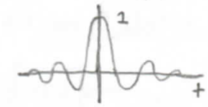


Rect function: $rect(t)$ $\begin{cases} 1, & 0 < t < 1 \\ 0, & \text{otherwise} \end{cases}$

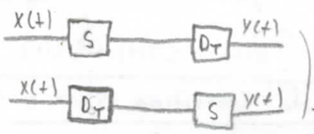


Impulse Train: $\delta_{T_s}(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT_s)$

Sinc Function: $\text{Sinc}(t) = \frac{\sin(\pi t)}{\pi t}$



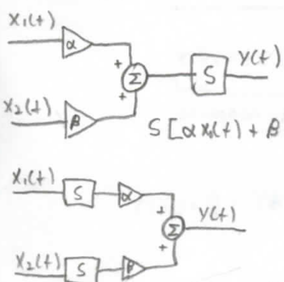
Time Invariance:



equivilant for Time-Invariance

Systems

Linearity:



equivilant for Linearity

Convolution Integral

$$f(t) * g(t) = g(t) * f(t)$$

$$\int_{-\infty}^{\infty} f(t-\tau)g(\tau)d\tau = \int_{-\infty}^{\infty} g(t-\tau)f(\tau)d\tau$$

Causal System

$h(t) = 0,\ t < 0$ "impulse response"

BIBO Stability

$$\int_{-\infty}^{\infty} |h(t)|dt < \infty$$

Asymptotic Stability

$Re(\lambda_i) < 0$ "Poles of transfer function"

Frequencies

$\omega = 2\pi f = \frac{2\pi}{T}$

Signal Operations

Scaling: $x(t) \xrightarrow{\alpha} \alpha x(t)$

Adder: $x_1(t)$, $x_2(t) \to \Sigma \to x_1(t) + x_2(t)$

Differentiator: $x(t) \to \frac{d}{dt} \to x'(t)$; $X(s) \to sX(s)$

Integrator: $x(t) \to \int dt \to \int x(t)dt$; $X(s) \to \frac{X(s)}{s}$

Time Shift: $x(t) \to D_T \to x(t-T)$

$x(t-T)$ DELAY

$x(t+T)$ ADVANCE

Time Scaling: $x(t) \to$ Time Scaling $\to x(at)$

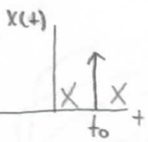
$|a| < 1$ STRETCH

$|a| > 1$ COMPRESS

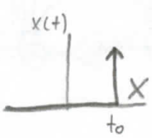
$a = (-)$ REFLECT

System Properties

① **Memoryless:** "y(t) at t_0 depends on x(t) at t_0"



② **Causality:** "y(t) at t_0 depends on x(t) at t_0 and the past but not future"



Fourier Series

$$x(t) = \sum_{k=-\infty}^{\infty} X_k e^{jk\omega_0 t}$$

$$X_k = \frac{1}{T}\int_T x(t)e^{-jk\omega_0 t} dt$$

Power (FS)

$$P_x = \frac{1}{T}\int_T |x(t)|^2 dt$$

$$P_x = \sum_{k=-\infty}^{\infty} |X_k|^2$$

Trigonometric Representations

$$x(t) = X_0 + 2\sum_{k=1}^{\infty} |X_k|\cos(k\omega_0 t + \theta_k)$$

$$= C_0 + 2\sum_{k=1}^{\infty} [C_k\cos(k\omega_0 t) + d_k\sin(k\omega_0 t)]$$

$Re(X_k) = C_k = \frac{1}{T}\int_T x(t)\cos(k\omega_0 t)dt$

$Im(X_k) = d_k = \frac{1}{T}\int_T x(t)\sin(k\omega_0 t)dt$

$|X_k| = \sqrt{C_k^2 + d_k^2}$

$\theta_k = -\tan^{-1}\left(\frac{d_k}{C_k}\right)$

*Note: $C_0 = X_0$ (DC component)

Frequency Response

EigenFunction Property, if the input is a complex exponential (or sinusoid), then $y_{ss}(t)$ (steady state) is given by: (input $x(t) = \alpha + \cos(\omega_0 t + \theta)$)

① $y_{ss}(t) = |H(j(0))|\cdot\alpha + |H(j\omega_0)|\cos(\omega t + \theta + \angle H(j\omega_0))$

② $x(t) = e^{j\omega_0 t}$: $y_{ss}(t) = |H(j(0))| + |H(j\omega_0)|e^{j(\omega t + \angle H(j\omega_0))}$

MT2 Material

Fourier Transform

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{j\omega t}dt \quad \text{IFT}$$

$$X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt \quad \text{FT}$$

Fourier Transform From Laplace Transform

"If the ROC of $X(s) = L[x(t)]$ contains the $j\omega$ axis, then the FT of $x(t)$ is given by: $X(\omega) = X(s)|_{s=j\omega}$"

Fourier Transform from Fourier Series

"If $x(t)$ is a periodic signal with fourier series, $X(\omega) = \sum_k 2\pi X_k\delta(\omega - k\omega_0)$"

Energy (FT)

$$E_x = \frac{1}{2\pi}\int_{-\infty}^{\infty} |X(\omega)|^2 d\omega$$

*Fourier Series from Laplace Transform

$X_k = \frac{1}{T_0}L[x(t)]\big|_{s=jk\omega_0}$ where: T_0 is the fundamental period

Nyquist

$\boxed{\omega_s > 2\omega_m}$

$f_s = \frac{1}{T_s} = \frac{\omega_s}{2\pi}$

Sampled Signal

$s(t) = x(t)p(t) \xleftrightarrow{FT} S(\omega) = \frac{1}{2\pi}X(\omega) * P(\omega)$

$$S(\omega) = \frac{1}{T_s}\sum_{k=-\infty}^{\infty} X(\omega - k\omega_s)$$

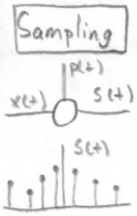
$$S(\omega) = \frac{1}{T_s}(\ldots X(\omega + \omega_s) + X(\omega) + X(\omega - \omega_s)\ldots)$$

Sampling

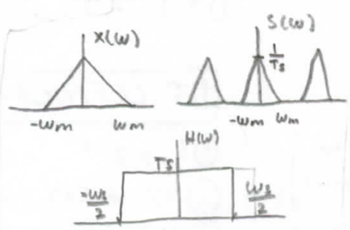
$p(t) = \sum_{k=-\infty}^{\infty}\delta(t - kT_s)$

$\updownarrow$ FT

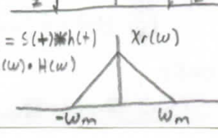
$\frac{2\pi}{T_s}\sum_{k=-\infty}^{\infty}\delta(\omega - k\omega_s)$

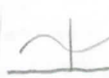


Reconstructing a Signal

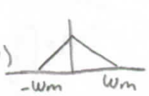


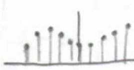
$X_r(\omega) = S(t) * h(t) = S(\omega)\cdot H(\omega)$



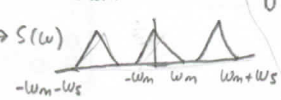


$x(t) \longleftrightarrow X(\omega)$





$s(t) \longleftrightarrow S(\omega)$



Periods / Frequencies

$\omega \to$ GCF (as big as possible)

$T \to$ LCM (as small as possible)

CT → use ω (Frequencies)

DT → use T (Periods, must be a whole number)

Cole Shanks
54950860

Discrete time signals and systems

Energy / Power:

$$E_x = \sum_{n=-\infty}^{\infty} |x[n]|^2$$

$$P_x = \lim_{N\to\infty} \frac{1}{2N+1}\sum_{n=-N}^{N} |x[n]|^2$$

BIBO stability:

$$\sum_k |h[k]| < \infty$$

Asymptotic stability:

$|\lambda_i| < 0$ "Poles of transfer function within unit circle"

Convolution Sum:

$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k]$$

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k] = \sum_{k=-\infty}^{\infty} h[k]x[n-k]$$

Causality

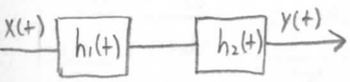
Signal: $x[n]$ is causal if $x[n]=0,\ n<0$

System: The system is causal if $h[n]=0,\ n<0$

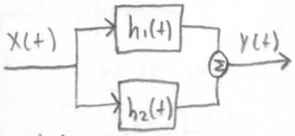
Causal LTI System:

$$y[n] = \sum_{k=0}^{n} x[k]h[n-k],\ n\ge 0$$

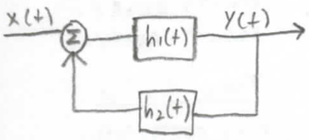
LTI System Connections



Cascade: $H(s) = H_1(s)\cdot H_2(s)$

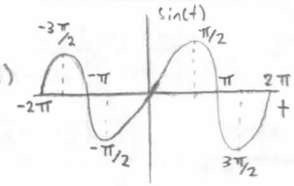


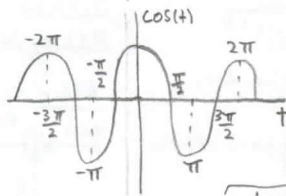
Parallel: $H(s) = H_1(s) + H_2(s)$



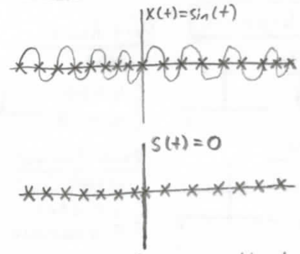
Feedback: $H(s) = \dfrac{H_1(s)}{1+H_2(s)\cdot H_1(s)}$

Sine / Cosine Graphs



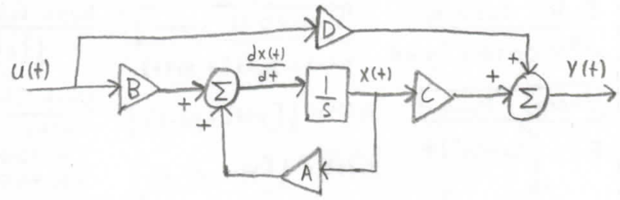


Problem with $\omega_s = 2\omega_m$



"Not aliased but amplitude reduced. Extreme @ sin(t). Ok @ cos(t)"

CT System State-Space LTI Model



$$\frac{dx(t)}{dt} = Ax(t) + Bu(t) \quad \text{State Differential eqn}$$

($u(t)$ → input)

output: $y(t) = Cx(t) + Du(t)$ Output eqn

State Transition Matrix (e^{At})

$$x(t) = \underbrace{e^{At}\cdot x_0}_{\text{zero-input response}} + \underbrace{\int_0^t e^{A(t-\tau)}Bu(\tau)\,d\tau}_{\text{zero-state response}}$$

$$\Big\updownarrow L\{x(t)\}$$

$$X(s) = (sI-A)^{-1}x_0 + (sI-A)^{-1}\cdot B\cdot U(s)$$

$$y(t) = \underbrace{Ce^{At}\cdot x_0}_{\text{zero-input response}} + \overbrace{C\int_0^t e^{A(t-\tau)}\cdot B\cdot u(\tau)\,d\tau + D\cdot u(\tau)}^{\text{zero-state response}}$$

$$\Big\updownarrow L\{y(t)\}$$

$$Y(s) = C(sI-A)^{-1}x_0 + \underbrace{[C(sI-A)^{-1}\cdot B + D]}_{H(s)\to \text{T.F Matrix}}U(s)$$

Controllability Matrix

$$x(t_f) = e^{At}x_0 + \int_0^{t_f} e^{A(t_f-\tau)}Bu(\tau)\,d\tau$$

$$M_c = \left[B, AB, A^2B, A^{N-1}B\right]$$

$\det(M_c) \neq 0 \rightarrow$ controllable

Observability Matrix

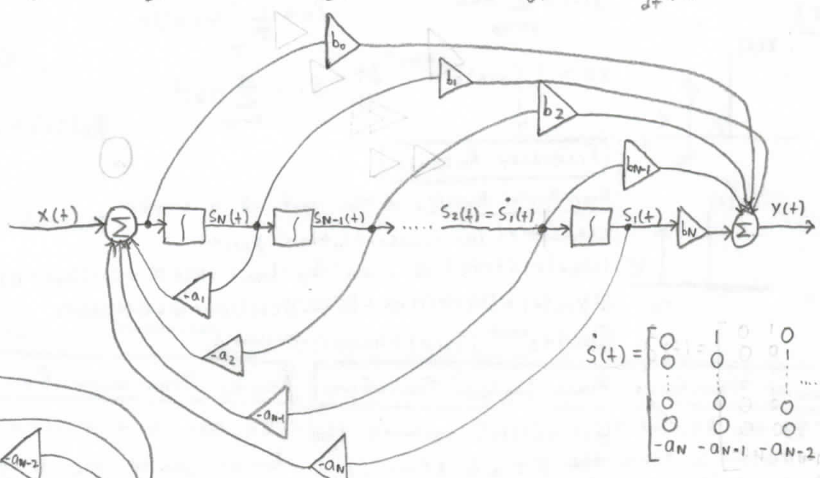
$$y(t) = Ce^{At}\cdot x_0 + C\int_0^t e^{A(t-\tau)}Bu(\tau)\,d\tau + Du(\tau)$$

$$M_o = \begin{bmatrix} C \\ CA \\ CA^2 \\ CA^{N-1}\end{bmatrix}$$

$\det(M_o) \neq 0 \rightarrow$ Observable

SISO Controllable Canonical Form

ODE: $\dfrac{d^N y(t)}{dt^N} + a_1\dfrac{d^{N-1}y(t)}{dt^{N-1}} + a_2\dfrac{d^{N-2}y(t)}{dt^{N-2}} \dots + a_{N-1}\dfrac{dy(t)}{dt} + a_N y(t) = b_0\dfrac{d^N x(t)}{dt^N} + b_1\dfrac{d^{N-1}x(t)}{dt^{N-1}} \dots + b_{N-1}\dfrac{dx(t)}{dt} + b_N x(t)$



$$\dot{s}(t) = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ -a_N & -a_{N-1} & -a_{N-2} & \cdots & -a_2 & -a_1 \end{bmatrix} s(t) + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} x(t)$$

$$y(t) = \left[(b_N - a_N b_0)\ (b_{N-1} - a_{N-1}b_0) \cdots (b_2 - a_2 b_0)\ (b_1 - a_1 b_0)\right] s(t) + b_0 x(t)$$

Diagonalization of a matrix

$A = TDT^{-1}$

$D = T^{-1}AT$

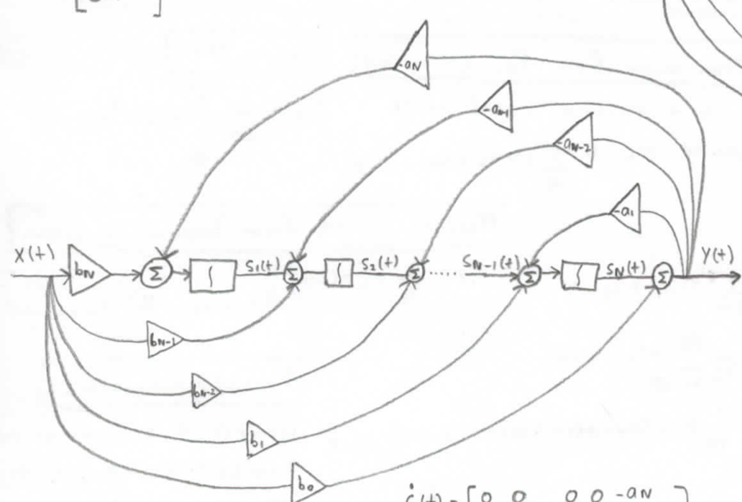
where: $D = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$ *eigenvalues of A

$T = [v_1, v_2]$ *eigenvectors of A

DT CCF/OCF

1. $\int \rightarrow z^{-1}$
2. $x(t) \rightarrow x[n]$, $y(t) \rightarrow y[n]$
3. $\dot{s}(t) = s[n+1]$

ODE: $y[n] + a_1 y[n-1] \dots + a_N y[n-N] = b_0 x[n] + b_1 x[n-1] \dots + b_N x[n-N]$



SISO Observable Canonical Form

ODE: (same as CCF)

$$\dot{s}(t) = \begin{bmatrix} 0 & 0 & \cdots & 0 & 0 & -a_N \\ 1 & 0 & \cdots & 0 & 0 & -a_{N-1} \\ 0 & 1 & \cdots & 0 & 0 & -a_{N-2} \\ \vdots & \vdots & & & & \vdots \\ 0 & 0 & \cdots & 1 & 0 & -a_2 \\ 0 & 0 & \cdots & 0 & 1 & -a_1 \end{bmatrix} s(t) + \begin{bmatrix} b_N - a_N b_0 \\ b_{N-1} - a_{N-1}b_0 \\ \vdots \\ b_2 - a_2 b_0 \\ b_1 - a_1 b_0 \end{bmatrix} x(t)$$

$$y(t) = [0\ 0 \cdots 0\ 1]\, s(t) + b_0 x(t)$$

Cole Shanks 54950860

Table 4.1 Basic Properties of Fourier Series

Basic Properties of Fourier Series		
	Time Domain	Frequency Domain
Signals and constants	$x(t)$, $y(t)$ periodic with period T_0, α, β	X_k, Y_k
Linearity	$\alpha x(t)+\beta y(t)$	$\alpha X_k+\beta Y_k$
Parseval's power relation	$P_x=\frac{1}{T_0}\int_{T_0}\lvert x(t)\rvert^2dt$	$P_x=\sum_k\lvert X_k\rvert^2$
Differentiation	$\frac{dx(t)}{dt}$	$jk\Omega_0X_k$
Integration	$\int_{-\infty}^{t}x(t')dt'$ only if $X_0=0$	$\frac{X_k}{jk\Omega_0}$ $k\neq 0$
Time shifting	$x(t-\alpha)$	$e^{-j\alpha\Omega_0}X_k$
Frequency shifting	$e^{jM\Omega_0t}x(t)$	X_{k-M}
Symmetry	$x(t)$ real	$\lvert X_k\rvert=\lvert X_{-k}\rvert$ even function of k; $\angle X_k=-\angle X_{-k}$ odd function of k
Convolution in time	$z(t)=[x*y](t)$	$Z_k=X_kY_k$

Table 3.2 One-sided Laplace Transforms

	Function of time	Function of s, ROC
(1)	$\delta(t)$	1, whole $s-$plane
(2)	$u(t)$	$\frac{1}{s}$, $\mathcal{R}e[s]>0$
(3)	$r(t)$	$\frac{1}{s^2}$, $\mathcal{R}e[s]>0$
(4)	$e^{-at}u(t), a>0$	$\frac{1}{s+a}$, $\mathcal{R}e[s]>-a$
(5)	$\cos(\Omega_0t)u(t)$	$\frac{s}{s^2+\Omega_0^2}$, $\mathcal{R}e[s]>0$
(6)	$\sin(\Omega_0t)u(t)$	$\frac{\Omega_0}{s^2+\Omega_0^2}$, $\mathcal{R}e[s]>0$
(7)	$e^{-at}\cos(\Omega_0t)u(t), a>0$	$\frac{s+a}{(s+a)^2+\Omega_0^2}$, $\mathcal{R}e[s]>-a$
(8)	$e^{-at}\sin(\Omega_0t)u(t), a>0$	$\frac{\Omega_0}{(s+a)^2+\Omega_0^2}$, $\mathcal{R}e[s]>-a$
(9)	$2Ae^{-at}\cos(\Omega_0t+\theta)u(t), a>0$	$\frac{A\angle\theta}{s+a-j\Omega_0}+\frac{A\angle-\theta}{s+a+j\Omega_0}$, $\mathcal{R}e[s]>-a$
(10)	$\frac{1}{(N-1)!}t^{N-1}u(t)$	$\frac{1}{s^N}$ N an integer, $\mathcal{R}e[s]>0$
(11)	$\frac{1}{(N-1)!}t^{N-1}e^{-at}u(t)$	$\frac{1}{(s+a)^N}$ N an integer, $\mathcal{R}e[s]>-a$
(12)	$\frac{2A}{(N-1)!}t^{N-1}e^{-at}\cos(\Omega_0t+\theta)u(t)$	$\frac{A\angle\theta}{(s+a-j\Omega_0)^N}+\frac{A\angle-\theta}{(s+a+j\Omega_0)^N}$, $\mathcal{R}e[s]>-a$

Table 3.1 Basic Properties of One-sided Laplace Transforms

Causal functions and constants	$\alpha f(t),\beta g(t)$	$\alpha F(s),\beta G(s)$
Linearity	$\alpha f(t)+\beta g(t)$	$\alpha F(s)+\beta G(s)$
Time shifting	$f(t-\alpha)u(t-\alpha)$	$e^{-\alpha s}F(s)$
Frequency shifting	$e^{\alpha t}f(t)$	$F(s-\alpha)$
Multiplication by t	$tf(t)$	$-\frac{dF(s)}{ds}$
Derivative	$\frac{df(t)}{dt}$	$sF(s)-f(0-)$
Second derivative	$\frac{d^2f(t)}{dt^2}$	$s^2F(s)-sf(0-)-f^{(1)}(0)$
Integral	$\int_{0-}^{t}f(t')dt'$	$\frac{F(s)}{s}$
Expansion/contraction	$f(\alpha t),\alpha\neq 0$	$\frac{1}{\lvert\alpha\rvert}F\left(\frac{s}{\alpha}\right)$
Initial value	$f(0-)=\lim_{s\to\infty}sF(s)$	

No.	$x_1(t)$	$x_2(t)$	$x_1(t)*x_2(t)=x_2(t)*x_1(t)$
1	$x(t)$	$\delta(t-T)$	$x(t-T)$
2	$e^{\lambda t}u(t)$	$u(t)$	$\frac{1-e^{\lambda t}}{-\lambda}u(t)$
3	$u(t)$	$u(t)$	$tu(t)$
4	$e^{\lambda_1 t}u(t)$	$e^{\lambda_2 t}u(t)$	$\frac{e^{\lambda_1 t}-e^{\lambda_2 t}}{\lambda_1-\lambda_2}u(t)$ $\lambda_1\neq\lambda_2$
5	$e^{\lambda t}u(t)$	$e^{\lambda t}u(t)$	$te^{\lambda t}u(t)$
6	$te^{\lambda t}u(t)$	$e^{\lambda t}u(t)$	$\frac{1}{2}t^2e^{\lambda t}u(t)$

L2.4 p177

Table 5.1 Basic Properties of Fourier Transform

	Time Domain	Frequency Domain
Signals and constants	$x(t),y(t),z(t),\alpha,\beta$	$X(\Omega),Y(\Omega),Z(\Omega)$
Linearity	$\alpha x(t)+\beta y(t)$	$\alpha X(\Omega)+\beta Y(\Omega)$
Expansion/contraction in time	$x(\alpha t)$, $\alpha\neq 0$	$\frac{1}{\lvert\alpha\rvert}X\left(\frac{\Omega}{\alpha}\right)$
Reflection	$x(-t)$	$X(-\Omega)$
Parseval's energy relation	$E_x=\int_{-\infty}^{\infty}\lvert x(t)\rvert^2dt$	$E_x=\frac{1}{2\pi}\int_{-\infty}^{\infty}\lvert X(\Omega)\rvert^2d\Omega$
Duality	$X(t)$	$2\pi x(-\Omega)$
Time differentiation	$\frac{d^nx(t)}{dt^n}, n\geq 1$, integer	$(j\Omega)^nX(\Omega)$
Frequency differentiation	$-jtx(t)$	$\frac{dX(\Omega)}{d\Omega}$
Integration	$\int_{-\infty}^{t}x(t')dt'$	$\frac{X(\Omega)}{j\Omega}+\pi X(0)\delta(\Omega)$
Time shifting	$x(t-\alpha)$	$e^{-j\alpha\Omega}X(\Omega)$
Frequency shifting	$e^{j\Omega_0t}x(t)$	$X(\Omega-\Omega_0)$
Modulation	$x(t)\cos(\Omega_ct)$	$0.5[X(\Omega-\Omega_c)+X(\Omega+\Omega_c)]$
Periodic signals	$x(t)=\sum_kX_ke^{jk\Omega_0t}$	$X(\Omega)=\sum_k2\pi X_k\delta(\Omega-k\Omega_0)$
Symmetry	$x(t)$ real	$\lvert X(\Omega)\rvert=\lvert X(-\Omega)\rvert$; $\angle X(\Omega)=-\angle X(-\Omega)$
Convolution in time	$z(t)=[x*y](t)$	$Z(\Omega)=X(\Omega)Y(\Omega)$
Windowing/Multiplication	$x(t)y(t)$	$\frac{1}{2\pi}[X*Y](\Omega)$
Cosine transform	$x(t)$ even	$X(\Omega)=\int_{-\infty}^{\infty}x(t)\cos(\Omega t)dt$, real
Sine transform	$x(t)$ odd	$X(\Omega)=-j\int_{-\infty}^{\infty}x(t)\sin(\Omega t)dt$, imaginary

Table 5.2 Fourier Transform Pairs

	Function of Time	Function of Ω
(1)	$\delta(t)$	1
(2)	$\delta(t-\tau)$	$e^{-j\Omega\tau}$
(3)	$u(t)$	$\frac{1}{j\Omega}+\pi\delta(\Omega)$
(4)	$u(-t)$	$\frac{-1}{j\Omega}+\pi\delta(\Omega)$
(5)	$\text{sign}(t)=2[u(t)-0.5]$	$\frac{2}{j\Omega}$
(6)	$A, -\infty<t<\infty$	$2\pi A\delta(\Omega)$
(7)	$Ae^{-at}u(t), a>0$	$\frac{A}{j\Omega+a}$
(8)	$Ate^{-at}u(t), a>0$	$\frac{A}{(j\Omega+a)^2}$
(9)	$e^{-a\lvert t\rvert}, a>0$	$\frac{2a}{a^2+\Omega^2}$
(10)	$\cos(\Omega_0t), -\infty<t<\infty$	$\pi[\delta(\Omega-\Omega_0)+\delta(\Omega+\Omega_0)]$
(11)	$\sin(\Omega_0t), -\infty<t<\infty$	$-j\pi[\delta(\Omega-\Omega_0)-\delta(\Omega+\Omega_0)]$
(12)	$p(t)=A[u(t+\tau)-u(t-\tau)], \tau>0$	$2A\tau\frac{\sin(\Omega\tau)}{\Omega\tau}$
(13)	$\frac{\sin(\Omega_0t)}{\pi t}$	$P(\Omega)=u(\Omega+\Omega_0)-u(\Omega-\Omega_0)$
(14)	$x(t)\cos(\Omega_0t)$	$0.5[X(\Omega-\Omega_0)+X(\Omega+\Omega_0)]$

Fourier Series of Discrete-time Periodic signals

	$x[n]$ periodic signal of period N	$X[k]$ periodic FS coefficients of period N
Z-transform	$x_1[n]=x[n](u[n]-u[n-N])$	$X[k]=\frac{1}{N}Z(x_1[n])\rvert_{z=e^{j2\pi k/N}}$
DTFT	$x[n]=\sum_kX[k]e^{j2\pi nk/N}$	$X(e^{j\omega})=\sum_k2\pi X[k]\delta(\omega-2\pi k/N)$
LTI response	input $x[n]=\sum_kX[k]e^{j2\pi nk/N}$	output: $y[n]=\sum_kX[k]H(e^{jk\omega_0})e^{j2\pi nk/N}$; $H(e^{j\omega})$ (frequency response of system)
Time-shift (circular shift)	$x[n-M]$	$X[k]e^{-j2\pi kM/N}$
Modulation	$x[n]e^{j2\pi Mn/N}$	$X[k-M]$
Multiplication	$x[n]y[n]$	$\sum_{m=0}^{N-1}X[m]Y[k-m]$ periodic convolution
Periodic convolution	$\sum_{m=0}^{N-1}x[m]y[n-m]$	$NX[k]Y[n]$

Properties of the DTFT

Z-transform:	$x[n], X(z), \lvert z\rvert=1\in ROC$	$X(e^{j\omega})=X(z)\rvert_{z=e^{j\omega}}$
Periodicity:	$x[n]$	$X(e^{j\omega})=X(e^{j(\omega+2\pi k)})$, k integer
Linearity:	$\alpha x[n]+\beta y[n]$	$\alpha X(e^{j\omega})+\beta Y(e^{j\omega})$
Time-shifting:	$x[n-N]$	$e^{-j\omega N}X(e^{j\omega})$
Frequency-shift:	$x[n]e^{j\omega_0n}$	$X(e^{j(\omega-\omega_0)})$
Convolution:	$(x*y)[n]$	$X(e^{j\omega})Y(e^{j\omega})$
Multiplication:	$x[n]y[n]$	$\frac{1}{2\pi}\int_{-\pi}^{\pi}X(e^{j\theta})Y(e^{j(\omega-\theta)})d\theta$
Symmetry:	$x[n]$ real-valued	$\lvert X(e^{j\omega})\rvert$ even function of ω; $\angle X(e^{j\omega})$ odd function of ω
Parseval's relation:	$\sum_{n=\infty}^{\infty}\lvert x[n]\rvert^2=\frac{1}{2\pi}\int_{-\pi}^{\pi}\lvert X(e^{j\omega})\rvert^2d\omega$	

Cole Shanks 54950860

One-sided Z-transforms

	Function of Time	Function of z, ROC
(1)	$\delta[n]$	1, Whole z-plane
(2)	$u[n]$	$\frac{1}{1-z^{-1}}$, $\lvert z\rvert > 1$
(3)	$nu[n]$	$\frac{z^{-1}}{(1-z^{-1})^2}$, $\lvert z\rvert > 1$
(4)	$n^2u[n]$	$\frac{z^{-1}(1+z^{-1})}{(1-z^{-1})^3}$, $\lvert z\rvert > 1$
(5)	$\alpha^n u[n]$, $\lvert\alpha\rvert < 1$	$\frac{1}{1-\alpha z^{-1}}$, $\lvert z\rvert > \lvert\alpha\rvert$
(6)	$n\alpha^n u[n]$, $\lvert\alpha\rvert < 1$	$\frac{\alpha z^{-1}}{(1-\alpha z^{-1})^2}$, $\lvert z\rvert > \lvert\alpha\rvert$
(7)	$\cos(\omega_0 n)u[n]$	$\frac{1-\cos(\omega_0)z^{-1}}{1-2\cos(\omega_0)z^{-1}+z^{-2}}$, $\lvert z\rvert > 1$
(8)	$\sin(\omega_0 n)u[n]$	$\frac{\sin(\omega_0)z^{-1}}{1-2\cos(\omega_0)z^{-1}+z^{-2}}$, $\lvert z\rvert > 1$
(9)	$\alpha^n\cos(\omega_0 n)u[n]$, $\lvert\alpha\rvert < 1$	$\frac{1-\alpha\cos(\omega_0)z^{-1}}{1-2\alpha\cos(\omega_0)z^{-1}+\alpha^2z^{-2}}$, $\lvert z\rvert > 1$
(10)	$\alpha^n\sin(\omega_0 n)u[n]$, $\lvert\alpha\rvert < 1$	$\frac{\alpha\sin(\omega_0)z^{-1}}{1-2\alpha\cos(\omega_0)z^{-1}+\alpha^2z^{-2}}$, $\lvert z\rvert > \lvert\alpha\rvert$

Table 10.2 Basic Properties of One-sided Z-transform

Causal signals and constants	$\alpha x[n]$, $\beta y[n]$	$\alpha X(z)$, $\beta Y(z)$
Linearity	$\alpha x[n]+\beta y[n]$	$\alpha X(z)+\beta Y(z)$
Convolution sum	$(x*y)[n]=\sum_k x[n]y[n-k]$	$X(z)Y(z)$
Time shifting – causal	$x[n-N]$ N integer	$z^{-N}X(z)$
Time shifting – non-causal	$x[n-N]$ $x[n]$ non-causal, N integer	$z^{-N}X(z)+x[-1]z^{-N+1}$ $+x[-2]z^{-N+2}+\cdots+x[-N]$
Time reversal	$x[-n]$	$X(z^{-1})$
Multiplication by n	$nx[n]$	$-z\frac{dX(z)}{dz}$
Multiplication by n^2	$n^2x[n]$	$z^2\frac{d^2X(z)}{dz^2}+z\frac{dX(z)}{dz}$
Finite difference	$x[n]-x[n-1]$	$(1-z^{-1})X(z)-x[-1]$
Accumulation	$\sum_{k=0}^{n}x[k]$	$\frac{X(z)}{1-z^{-1}}$
Initial value	$x[0]$	$\lim_{z\to\infty}X(z)$
Final value	$\lim_{n\to\infty}x[n]$	$\lim_{z\to 1}(z-1)X(z)$

Discrete-time Fourier Transforms (DTFT)

	Discrete-time signal	DTFT $X(e^{j\omega})$, periodic of period 2π
(1)	$\delta[n]$	$1, -\pi\le\omega<\pi$
(2)	A	$2\pi A\delta(\omega), -\pi\le\omega<\pi$
(3)	$e^{j\omega_0 n}$	$2\pi\delta(W-\omega_0), -\pi\le\omega<\pi$
(4)	$\alpha^n u[n], \lvert\alpha\rvert<1$	$\frac{1}{1-\alpha e^{-j\omega}}, -\pi\le\omega<\pi$
(5)	$n\,\alpha^n u[n], \lvert\alpha\rvert<1$	$\frac{\alpha e^{-j\omega}}{(1-\alpha e^{-j\omega})^2}, -\pi\le\omega<\pi$
(6)	$\cos(\omega_0 n)u[n]$	$\pi[\delta(\omega-\omega_0)+\delta(\omega+\omega_0)], -\pi\le\omega<\pi$
(7)	$\sin(\omega_0 n)u[n]$	$-j\pi[\delta(\omega-\omega_0)+\delta(\omega+\omega_0)], -\pi\le\omega<\pi$
(8)	$\alpha^{\lvert n\rvert}, \lvert\alpha\rvert<1$	$\frac{1-\alpha^2}{1-2\alpha\cos(\omega)+\alpha^2}, -\pi\le\omega<\pi$
(9)	$p[n]=u[n+N/2]-u[n-N/2]$	$\frac{\sin(\omega(N+1)/2)}{\sin(\omega/2)}, -\pi\le\omega<\pi$
(10)	$\alpha^n\cos(\omega_0 n)u[n]$	$\frac{1-\alpha\cos(\omega_0)e^{-j\omega}}{1-2\alpha\cos(\omega_0)e^{-j\omega}+\alpha^2e^{-2j\omega}}, -\pi\le\omega<\pi$
(11)	$\alpha^n\sin(\omega_0 n)u[n]$	$\frac{\alpha\sin(\omega_0)e^{-j\omega}}{1-2\alpha\cos(\omega_0)e^{-j\omega}+\alpha^2e^{-2j\omega}}, -\pi\le\omega<\pi$

$\alpha^{n-1}u[n-1] \leftrightarrow \frac{e^{-jw}}{1-\alpha e^{-jw}}$

Discrete Fourier Transform (DFT) (Fourier Series Coefficients)

	$x[n]$ finite-length N aperiodic signal	$\tilde{x}[n]$ periodic extension of period $L\ge N$
	$\tilde{x}[n]=\frac{1}{N}\sum_{k=0}^{L-1}\tilde{X}[k]e^{j2\pi nk/L}$	$\tilde{X}[k]=\sum_{n=0}^{L-1}\tilde{x}[n]e^{-j2\pi nk/L}$
IDFT/DFT	$x[n]=\tilde{x}[n]W[n]$, $W[n]=u[n]-u[n-N]$	$X[k]=\tilde{X}[k]W[n]$, $W[k]=u[k]-u[k-N]$
Circular convolution	$(x\otimes_L y)[n]$	$X[k]Y[k]$
Circular and linear convolution	$(x\otimes_L y)[n]=(x*y)[n]$, $L\ge M+K-1$ $M=$ length of $x[n]$, $K=$ length of $y[n]$	

$$\sum_{n=0}^{N} ar^n = \frac{a(1-r^{N+1})}{1-r}$$

No.	$f(t)$	$\mathcal{L}(f)(s)=F(s)$		Reference
1.	1	$\frac{1}{s}$,	$s>0$	Equation (1.5)
2.	t^n	$\frac{n!}{s^{n+1}}$,	$s>0$	Equation (1.8)
3.	$\sin at$	$\frac{a}{s^2+a^2}$,	$s>0$	Example 1.9
4.	$\cos at$	$\frac{s}{s^2+a^2}$,	$s>0$	Equation (1.10)
5.	e^{at}	$\frac{1}{s-a}$,	$s>a$	Example 1.4
6.	$e^{at}\sin bt$	$\frac{b}{(s-a)^2+b^2}$,	$s>a$	Prop. 2.12 with $f=\sin bt$
7.	$e^{at}\cos bt$	$\frac{s-a}{(s-a)^2+b^2}$,	$s>a$	Prop. 2.12 with $f=\cos bt$
8.	t^ne^{at}	$\frac{n!}{(s-a)^{n+1}}$,	$s>a$	Prop. 2.14 with $f=e^{at}$

Euler's formula $\quad e^{j\theta}=\cos(\theta)+j\sin(\theta)$

…for cosine $\quad \cos(\theta)=\dfrac{e^{j\theta}+e^{-j\theta}}{2}$

…for sine $\quad \sin(\theta)=\dfrac{e^{j\theta}-e^{-j\theta}}{2j}$

sinc function $\quad \mathrm{sinc}(\theta):=\dfrac{\sin(\pi\theta)}{\pi\theta}$

6. Sum-Difference Formulas

$\sin(x+y)=\sin x\cos y+\cos x\sin y$ $\quad \tan(x+y)=\frac{\tan x+\tan y}{1-\tan x\tan y}$

$\sin(x-y)=\sin x\cos y-\cos x\sin y$

$\cos(x+y)=\cos x\cos y-\sin x\sin y$ $\quad \tan(x-y)=\frac{\tan x-\tan y}{1+\tan x\tan y}$

$\cos(x-y)=\cos x\cos y+\sin x\sin y$

7. Double Angle Formulas

$\sin(2x)=2\sin x\cos x$

$\cos(2x)=\cos^2x-\sin^2x=2\cos^2x-1=1-2\sin^2x$

$\tan(2x)=\frac{2\tan x}{1-\tan^2x}$

8. Power-Reducing/Half Angle Formulas

$\sin^2x=\frac{1-\cos 2x}{2}$ $\quad \cos^2x=\frac{1+\cos 2x}{2}$ $\quad \tan^2x=\frac{1-\cos 2x}{1+\cos 2x}$

$\sin\frac{x}{2}=\pm\sqrt{\frac{1-\cos x}{2}}$ $\quad \cos\frac{x}{2}=\pm\sqrt{\frac{1+\cos x}{2}}$ $\quad \tan\frac{x}{2}=\pm\sqrt{\frac{1-\cos x}{1+\cos x}}=\frac{\sin x}{1+\cos x}=\frac{1-\cos x}{\sin x}$

$\boxed{\text{Recap of Transforms}}$ DT

$LT: X(s)=\int_{-\infty}^{\infty}x(t)e^{-st}dt$ $\quad ZT: X(z)=\sum_{n=-\infty}^{\infty}x[n]z^{-n}$

$ILT: x(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}X(s)e^{st}ds$ $\quad IZT: x[n]=\frac{1}{2\pi j}\oint X(z)z^{n-1}dz$

$FT: X(w)=\int_{-\infty}^{\infty}x(t)e^{-jwt}dt$ $\quad DTFT: X(w)=\sum_{n=-\infty}^{\infty}x[n]e^{-jwn}$

$IFT: x(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}X(w)e^{jwt}dw$ $\quad IDTFT: x[n]=\frac{1}{2\pi}\int_{-\pi}^{\pi}X(w)e^{jwn}dw$

$FS: X_k=\frac{1}{T}\int_T x(t)e^{-jkw_0t}dt$ $\quad DFT: X_k=\frac{1}{N}\sum_{n=0}^{N-1}x[n]e^{-jkw_0n}$

$IFS: x(t)=\sum_{k=-\infty}^{\infty}X_ke^{jkw_0t}$ $\quad IDFT: x[n]=\sum_{k=0}^{N-1}X_ke^{jkw_0n}$

Where $w_0=\frac{2\pi}{N}$